Problem A.21

It's obvious that the trace of a *diagonal* matrix is the sum of its eigenvalues, and its determinant is their product (just look at Equation A.79). It follows (from Equations A.65 and A.68) that the same holds for any *diagonalizable* matrix. Prove that in fact

$$\det(\mathsf{T}) = \lambda_1\lambda_2\cdots\lambda_n, \quad \mathrm{Tr}(\mathsf{T}) = \lambda_1 + \lambda_2 + \cdots + \lambda_n, \tag{A.93}$$

for *any* matrix. (The λ's are the n solutions to the characteristic equation—in the case of multiple roots, there may be fewer linearly-independent eigen*vectors* than there are solutions, but we still count each λ as many times as it occurs.) *Hint:* Write the characteristic equation in the form

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) = 0,$$

and use the result of Problem A.20.

Solution

An $n \times n$ matrix representing a linear transformation $\hat{T}$ has the general form,

$$\mathsf{T} = \begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & \cdots & \cdots & T_{nn} \end{bmatrix},$$

in a basis. The eigenvalue problem it satisfies is

$$\mathsf{Ta} = \lambda\mathsf{a},$$

which implies that

$$\det(\mathsf{T} - \lambda\mathsf{I}) = 0$$

$$\begin{vmatrix} T_{11} - \lambda & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} - \lambda & \cdots & T_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ T_{n1} & T_{n2} & \cdots & T_{nn} - \lambda \end{vmatrix} = 0.$$

This determinant is in general an nth-degree polynomial.

$$C_n\lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_1\lambda + C_0 = 0$$

By the fundamental theorem of algebra, there exist n roots to this equation: λ_1, λ_2, ..., λ_n. As a result, the equation may be written as

$$(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n) = 0.$$

Multiply both sides by $(-1)^n$.

$$(\lambda_1 - \lambda)(\lambda_2 - \lambda)\cdots(\lambda_n - \lambda) = 0$$

Multiply the terms on the left together.

$$\begin{aligned}
0 &= (\lambda_1\lambda_2\cdots\lambda_n) + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n \text{ times}} + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}}\lambda_n + \cdots + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}}\lambda_2 \\
&\qquad + \overbrace{(-\lambda)(-\lambda)\cdots(-\lambda)}^{n-1 \text{ times}}\lambda_1 + \cdots \\
&= (-\lambda)^n + (-\lambda)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) + \cdots + \lambda_1\lambda_2\cdots\lambda_n \\
&= (-1)^n\lambda^n + (-1)^{n-1}\lambda^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n) + \cdots + \lambda_1\lambda_2\cdots\lambda_n \\
&= \underbrace{(-1)^n}_{C_n}\lambda^n + \underbrace{(-1)^{n-1}(\lambda_1 + \lambda_2 + \cdots + \lambda_n)}_{C_{n-1}}\lambda^{n-1} + \cdots + \underbrace{\lambda_1\lambda_2\cdots\lambda_n}_{C_0}
\end{aligned}$$

According to Problem A.20,

$$\begin{aligned}
C_n &= (-1)^n \\
C_{n-1} &= (-1)^{n-1}\,\mathrm{Tr}(\mathsf{T}) \\
C_0 &= \det(\mathsf{T}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathrm{Tr}(\mathsf{T}) &= \lambda_1 + \lambda_2 + \cdots + \lambda_n \\
\det(\mathsf{T}) &= \lambda_1\lambda_2\cdots\lambda_n.
\end{aligned}$$